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Partition Theorems for Abelian Groups

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Let A be a finite matrix with integral entries and G be an Abelian group. Define A to be *partition regular* in G if for every partition of $G\backslash\{0\}$ into finitely many classes there exist elements $x_1, \dots, x_m$ contained in one class such that $A(x_1, \dots, x_m)^T = 0$. THEOREM. *A is partition regular in G iff at least one of the following statements holds.* (*i*) *There is* $x \in G\backslash\{0\}$ *such that* $A(x, \dots, x)^T = 0$. (ii) *A is partition regular in* $Z_p^{\aleph_0}$ (*p prime*) *and* $Z_p^{\aleph_0} \subset G$. (iii) *A is partition regular in Z and the set of orders of elements in G is unbounded.*

1. INTRODUCTION

In 1917 Schur [12] proved the following theorem. If the nonzero integers are partitioned into finitely many classes then there exist integers x_1, x_2, x_3 contained in one class such that

$$x_1 + x_2 = x_3 .$$

This was generalized by Rado [9] to the following theorem.

THEOREM. *Let n be a positive integer. If the nonzero integers are partitioned into finitely many classes then there exist integers* $x_1, \dots, x_n$ *such that all the sums*

$$\sum_1^n \epsilon^i x_i \qquad (\textit{where } \epsilon \in \{0, 1\} \textit{ and not all } \epsilon^i = 0)$$

are in the same class.

For $n = 3$ this theorem is equivalent to the following. If the nonzero

582a/19/1-7

integers are partitioned into finitely many classes then there are integers $x_1, \ldots, x_7$ contained in one class such that

$$\begin{pmatrix} 1, & 1, & 1, & -1, & 0, & 0, & 0 \\ 1, & 1, & 0, & 0, & -1, & 0, & 0 \\ 1, & 0, & 1, & 0, & 0, & -1, & 0 \\ 0, & 1, & 1, & 0, & 0, & 0, & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_7 \end{pmatrix} = \mathbf{0}.$$

From this formulation the following interesting question arose. For which finite matrices **A** with entries in Z (the set of all integers) is it true that for every partition of the nonzero integers into finitely many classes there exist $x_1, \ldots, x_m$ contained in one class such that

$$\mathbf{A} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \mathbf{0}?$$

This problem was completely solved by Rado [10, 11] and Deuber [2].

Here we give the complete solution of the above question for Abelian groups instead of integers. Let us define a matrix A with entries in Z to be partition regular in an Abelian group G if for every partition of $G \backslash \{0\}$ into finitely many classes at least one of the classes contains elements $x_1, \ldots, x_m$ such that

$$\mathbf{A} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \mathbf{0}.$$

It turns out that **A** is partition regular in G iff at least one of the following statements holds.

(i) **A** is partition regular in Z and the set of orders of elements of G is unbounded.

(ii) There is a prime p such that Z_p^ω (the $\aleph_0$-dimensional vectorspace over Z_p) is isomorphic to a subgroup of G and A is partition regular in Z_p^ω.

(iii) There exists $x \in G \backslash \{0\}$ such that

$$\mathbf{A} \cdot \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} = \mathbf{0}.$$

In [2] the subsets of Z which for every **A** partition regular in Z contain a solution of $\mathbf{AX} = \mathbf{0}$ are analyzed. Here the corresponding theory for

Abelian groups is developed as well. Let us finally remark that the restriction to Abelian groups is natural in the sense that for non-Abelian groups the systems of equations considered are no longer homogenous.

2. Definitions

Z, N, P are the sets of integers, positive integers, and positive primes. Let ∞ be an element not contained in Z and $P^* = P \cup \{\infty\}$. Throughout the paper p, q (possibly with subscripts) will denote elements of P^*.

$Z_{m,n}$ ($m, n \in N$) is the set of all matrices with entries in Z having m columns and n rows. $\mathbf{A} = (\mathfrak{a}^1, \ldots, \mathfrak{a}^m)$, $\mathbf{B} = (\mathfrak{b}^1, \ldots, \mathfrak{b}^m)$ denote matrices with the indicated columns. $\mathbf{0}$ denotes the zero matrix of any size.

Let $m, n \in N$ and $\mathbf{A} \in Z_{m,n}$. $\mathbf{A}$ has the ∞-column property or column property if there exists a partition $f: \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that:

$$\sum_{f(i)=1} \mathfrak{a}^i = \mathbf{0}$$

and furthermore for all $j \in \{2, \ldots, m\}$ there exist $b_0{}^j \in N$, $b_i{}^j \in Z$, $(f(i) \in \{1, \ldots, j-1\})$, such that

$$\sum_{f(i)=j} \mathfrak{a}^i b_0{}^j + \sum_{f(i)<j} \mathfrak{a}^i b_i{}^j = \mathbf{0}.$$

$\mathbf{A}$ has the p-column property ($p \in P$) if the above equations hold modulo p and $b_0{}^j \neq 0 \pmod p$, ($j \in (2, \ldots, m\}$). Let $\Gamma(\mathbf{A}) = \{p \in P^* \mid \mathbf{A}$ has p-column property$\}$.

Z_r ($r \in N$) is the cyclic group with r elements, and $Z_\infty = Z$. By a group we mean an Abelian group. Let G be a group and $x \in G$; ord x is the smallest $n \in N$ such that $nx = 0$. If no such n exists then let ord $x = \infty$. Let G, H be groups; $G \oplus H$ is the direct sum of G and H. G^n is the n-fold direct sum of G's and G^ω is the $\aleph_0$-direct sum of G's.

Let $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\} \subset Z_p^\omega$, ($m \in N$, $p \in P$). $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)$ is an m-basis if $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\}$ is linearly independent in Z_p^ω.

A vector $\mathfrak{x} = \sum_1^m \lambda_i \mathfrak{x}_i \in Z_p^\omega \setminus \{0\}$, ($\lambda_i \in \{0, \ldots, p-1\}$) is a standard vector with respect to the m-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)$ if $\lambda_{i_0} = 1$, where $i_0 = \min\{i \mid \lambda_i \neq 0\}$. Let $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)_p$ be the set of all standard vectors with respect to $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)$. A subset X of Z_p^ω is an m-set if $X = (\mathfrak{x}_1, \ldots, \mathfrak{x}_m)_p$ for some m-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)$. Note that for every $\mathfrak{x} \in Z_p^\omega \setminus \{0\}$ the singleton $\{\mathfrak{x}\}$ is a 1-set. An n-set $(\mathfrak{y}_1, \ldots, \mathfrak{y}_n)_p$ is an n-subset of an m-set $(\mathfrak{x}_1, \ldots, \mathfrak{x}_m)_p$ if $(\mathfrak{y}_1, \ldots, \mathfrak{y}_n)_p \subset (\mathfrak{x}_1, \ldots, \mathfrak{x}_m)_p$. This defines a directed partial order on the set of all m-sets ($m \in N$).

Remark. Let $(\mathfrak{r}_j \mid j \in N)$ be a basis for Z_p^ω. For a vector $\mathfrak{r} = \sum \lambda_j \mathfrak{r}_j \in Z_p^\omega \backslash \{0\}$ let $i(\mathfrak{r}) = \min\{j \mid \lambda_j \neq 0\}$ be the leading index of $\mathfrak{r}$. Every m-dimensional subspace U of Z_p^ω has a basis $(\mathfrak{u}_1, ..., \mathfrak{u}_m)$ consisting of standard vectors with respect to $(\mathfrak{r}_j \mid j \in N)$ satisfying $i(\mathfrak{u}_1) < i(\mathfrak{u}_2) < \cdots < i(\mathfrak{u}_m)$.

Let $k \in N$ and X be a set. A k-partition of X is a mapping $f: X \to Y$ where Y is any k-element set. The sets $f^{-1}(y)$, $(y \in Y)$, are the classes of the partition; in particular $f^{-1}(y)$ is the class with index y.

Let $k, m, n \in N$, $\mathbf{A} \in Z_{m,n}$, and G be a group. $\mathbf{A}$ is k-partition regular in a (possibly nonproper) subset X of $G \backslash \{0\}$ if for every k-partition of $G \backslash \{0\}$ there exist $\mathfrak{r}_1, ..., \mathfrak{r}_m \in X$ all contained in the same class of the partition such that

$$\mathbf{AX} = \mathbf{0} \qquad (\mathbf{X} = (\mathfrak{r}_1, ..., \mathfrak{r}_m)^T).$$

Note that if $\mathbf{A}$ is k-regular in X then it is k'-regular in X for all $k' \leqslant k$. If $\mathbf{A}$ is k-regular in G then it is k-regular in every homomorphic preimage of G especially in every group isomorphic to G and in every group containing G.

$\mathbf{A}$ is partition regular in G if $\mathbf{A}$ is k-partition regular in G for all $k \in N$. Note that if $\mathbf{A}$ is partition regular in G then G is infinite, except for trivial cases.

3. Preliminary Results

The following compactness lemma is a direct consequence of König's infinity lemma or of Tychonoff's theorem.

Lemma 1. *Let $k, n, m \in N$, $\mathbf{A} \in Z_{m,n}$ and G be a group. $\mathbf{A}$ is k-partition regular in G if $\mathbf{A}$ is k-partition regular in some finite subset of G.*

Lemma 2. *Let $k, m, n \in N$, $\mathbf{A} \in Z_{m,n}$ and G, H be groups. If $\mathbf{A}$ is $2k$-partition regular in $G \oplus H$ then $\mathbf{A}$ is k-partition regular in G or $\mathbf{A}$ is k-partition regular in H.*

Proof 2. Assume that $\mathbf{A}$ is not k-partition regular in G. Hence there exists a k-partition f_1 of $G \backslash \{0\}$ such that no class of f_1 contains a solution of $\mathbf{AX} = \mathbf{0}$. Let f_2 be an arbitrary k-partition of $H \backslash \{0\}$. Define a $2k$-partition f_3 of $G \oplus H \backslash \{0\}$:

$$f_3(x + y) = \begin{cases} (1, f_1(x)) & \text{if } x \neq 0 \\ (2, f_2(y)) & \text{if } x = 0 \end{cases} \quad (x \in G, y \in H).$$

If $\mathbf{A}$ is $2k$-partition regular in $G \oplus H$ we have by definition of f_1 that every solution $x_1 + y_1, \ldots, x_m + y_m$ of $\mathbf{AX} = \mathbf{0}$ contained in one class of f_3 satisfies $x_1 = x_2 = \cdots = x_m = 0$ and $f_2(y_1) = \cdots = f_2(y_m)$. So $\mathbf{A}$ is k-partition regular in H.

Remark. The most interesting fact in Lemma 2 is that $2k$-partition regularity in $G \oplus H$ already implies k regularity in one of the summands. In general $2k$-partition regularity in the product of two structures does not imply k-regularity in one of the components.

COROLLARY 3. *Let $k, m, n \in N$, $\mathbf{A} \in Z_{m,n}$ and G, H be groups.* $\mathbf{A}$ *is partition regular in $G \oplus H$* iff *it is partition regular in G or partition regular in H.*

LEMMA 4 [9]. *Let $m, n \in N$, $\mathbf{A} \in Z_{m,n}$. There exists $p \in P$ such that* $\mathbf{A}$ *has the column property* iff $\mathbf{A}$ *has the p'-column property for all $p' \geqslant p$.*

Proof 4. If $\mathbf{A}$ has the column property then it has the p'-column property for all $p' > \max\{b_0^2, \ldots, b_0^m\}$.

On the other hand $\mathbf{A}$ has column property iff certain systems of linear equations with coefficients in $\mathbf{A}$ have a solution. Such a system of linear equations has a solution iff there exists a solution in terms of determinants of submatrices of $\mathbf{A}$. There are only finitely many such determinants. So the set of integers required in order to determine the column property of $\mathbf{A}$ is bounded. Hence for p large enough the calculations modulo p are absolute.

DEFINITION. Let $m, n \in N$, $\mathbf{A} \in Z_{m,n}$. Let $p_{\mathbf{A}}$ be the smallest $p \in P$ such that $\mathbf{A}$ has column property iff $\mathbf{A}$ has p-column property.

Note that either $\Gamma(\mathbf{A}) \subset \{2, \ldots, p_{\mathbf{A}} - 1\}$ or $\infty \in \Gamma(\mathbf{A})$.

4. PARTITION REGULARITY IN SPECIAL GROUPS

The following characterization of the matrices which are partition regular in Z is due to Rado [9].

THEOREM 5. *Let $m, n \in N$, $\mathbf{A} \in Z_{m,n}$.* $\mathbf{A}$ *is partition regular in Z* iff $\mathbf{A}$ *has the column property.*

Outline of proof. We need

DEFINITION. Let $p \in P$. For each $x \in P$ let $x = p^{\alpha(x)} \cdot (p \cdot k(x) + l(x))$ with $\alpha(x) \in N$, $k(x) \in Z$, $l(x) \in \{1, \ldots, p-1\}$. Then $R_p(x) = l(x)$ is a p-partition of $Z \backslash \{0\}$.

If $\mathbf{A}$ is partition regular in Z then $\mathbf{AX} = \mathbf{0}$ has a solution $x_i = p_{\mathbf{A}}^{\alpha_i}(p_{\mathbf{A}}k_i + l)$ contained in one class of $R_{p_{\mathbf{A}}}$. Collect all i's with the same α_i. This defines a partition on $\{1,..., m\}$. By evaluating $\sum a^i x_i = 0$ modulo certain powers of $p_{\mathbf{A}}$ it is established that $\mathbf{A}$ has the $p_{\mathbf{A}}$-column property. Hence $\mathbf{A}$ has the column property.

If $\mathbf{A}$ has the column property, then by an ingenious application of van der Waerden's theorem on arithmetical progressions [13] it is established that $\mathbf{A}$ is partition regular in Z.

As the $p_{\mathbf{A}}$-partition regularity of $\mathbf{A}$ in Z implies the $p_{\mathbf{A}}$-column property of $\mathbf{A}$, we obtain from Lemma 1:

COROLLARY 6. *Let* $\alpha, m, n \in N$, $\mathbf{A} \in Z_{m,n}$, $p \in P$, $p \geqslant p_{\mathbf{A}}$. *If* $\mathbf{A}$ *is* $p_{\mathbf{A}}$*-partition regular in* Z_{p^α} *then* $\mathbf{A}$ *has the column property.*

The following theorem improves Lemma 2 for $G = H = Z$.

THEOREM 7. *Let* $k, m, n \in N$, $\mathbf{A} \in Z_{m,n}$. *If* $\mathbf{A}$ *is* $p_{\mathbf{A}}$*-partition regular in* Z^k *then* $\mathbf{A}$ *has the column property.*

Proof 7. For $\mathfrak{x} = (x^1,..., x^k) \in Z^k \backslash \{0\}$ let $n_{\mathfrak{x}}$ be the smallest n such that $x^n \neq 0$. Define a $p_{\mathbf{A}}$-partition f of $Z^k \backslash \{0\}$ by $f(\mathfrak{x}) = R_{p_{\mathbf{A}}}(x^{n_{\mathfrak{x}}})$. Let $\mathfrak{x}_1 = (x_1^1,..., x_1^k),..., \mathfrak{x}_m = (x_m^1,..., x_m^k)$ be a solution of $\mathbf{AX} = \mathbf{0}$ contained in one class of f. Let $\{n_{\mathfrak{x}_i} \mid i = 1,..., m\} = \{n_1,..., n_t\}$ and $n_1 < n_2 < \cdots < n_t$. Let $\mathbf{A}_j$, $(j = 1,..., t)$, be the submatrix of $\mathbf{A}$ generated by the columns $\mathfrak{a}^i$ with $n_{\mathfrak{x}_i} \in \{n_1,..., n_j\}$. By induction on j it is established that all $\mathbf{A}_j$ have the $p_{\mathbf{A}}$-column property, from which it follows that all $\mathbf{A}_j$ have the column property.

In order to see this for $\mathbf{A}_1$, calculate the n_1th coordinate and obtain

$$\sum_{n_{\mathfrak{x}_i} = n_1} \mathfrak{a}^i x_i^{n_1} = \mathbf{0}. \tag{1}$$

By definition of f and $R_{p_{\mathbf{A}}}$ it follows that $\mathbf{A}_1$ has the $p_{\mathbf{A}}$-column property and hence $\mathbf{A}_1$ has the column property. Assume that $\mathbf{A}_{j-1}$ has the column property and calculate the n_jth coordinate, obtaining

$$\mathbf{0} = \sum_{n_{\mathfrak{x}_i} = n_j} \mathfrak{a}^i x_i^{n_j} + \sum_{n_{\mathfrak{x}_i} < n_j} \mathfrak{a}^i x_i^{n_j}. \tag{2}$$

By definition of f we have for all i with $n_{\mathfrak{x}_i} = n_j$:

$$x_i^{n_j} = p_{\mathbf{A}}^{\alpha(x_i^{n_j})}(p_{\mathbf{A}} k(x_i^{n_j}) + l).$$

Let $\{\alpha(x_i^{n_j}) \mid n_{\mathfrak{x}_i} = n_j\} = \{\alpha_1,..., \alpha_s\}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_s$. By evaluating (2) modulo $p_{\mathbf{A}}^{\alpha_1+1},..., p_{\mathbf{A}}^{\alpha_s+1}$ it is established that $\mathbf{A}_j$ has the $p_{\mathbf{A}}$-column property.

COROLLARY 8. *Let* $m, n, t, \alpha_1, \ldots, \alpha_t, r_1, \ldots, r_{t+1} \in N$, $q_1, \ldots, q_t \in P$, *and* $q_1, \ldots, q_t \geqslant p_{\mathbf{A}}$, $\mathbf{A} \in Z_{m,n}$. *If* $\mathbf{A}$ *is* $p_{\mathbf{A}}$*-partition regular in*

$$Z_{q_1^{\alpha_1}}^{r_1} \oplus \cdots \oplus Z_{q_t^{\alpha_t}}^{r_t} \oplus Z^{r_{t+1}}$$

then $\mathbf{A}$ *has the column property.*

Proof 8. The proof of Theorem 7 can be copied with the modification that Eqs. (1), (2) are to be considered modulo q_i for some $i \in \{1, \ldots, t\}$. But this does not matter as all the q's are large enough.

Next we describe the matrices which are partition regular in $Z_p{}^\omega (p \in P)$. We need the following theorem of Graham, Leeb, and Rothschild [4, 5].

THEOREM 9. *Let* $m, k \in N, p \in P$. *Then there exists* r *such that for every* k*-partition of the one-dimensional subspaces of* Z_p^r *there exists an* m*-dimensional subspace of* Z_p^r *with all its one-dimensional subspaces in the same class.*

COROLLARY 10. *Let* $k, m, n \in N, p \in P$. *For every* k*-partition* f *of* $Z_p^\omega \backslash \{0\}$ *there exist subspaces* U, V *of* Z_p^ω *with the following properties.* $U \cap V = \mathbf{0}$; $\dim U = m$, $\dim V = n$; $f(x+y) = f(x)$ *whenever*

$$x \in U \backslash \{0\}, y \in V \backslash \{0\}.$$

Proof 10. Let $(\mathfrak{r}_i \mid i \in N)$ be a basis for Z_p^ω. Let f be a k-partition of $Z_p^\omega \backslash \{0\}$.

Let f_1 be the following k^{p-1}-partition of the one-dimensional subspaces of $Z_p^\omega : f_1(X) = (f(\mathfrak{r}), f(2\mathfrak{r}), \ldots, f((p-1)\mathfrak{r}))$, where $\mathfrak{r}$ is the standard vector generating X. By Theorem 9 there exists an $(m+n)$-dimensional subspace of Z_p^ω with all its one-dimensional subspaces in the same class of f. Let $\mathfrak{r}^1, \ldots, \mathfrak{r}^{m+n}$ be a basis of standard vectors for this space and

$$\mathfrak{r}^j = \mathfrak{r}_{i_0^j} + \sum m_k^j \mathfrak{r}_{i_k^j} \qquad \text{and} \qquad i_0^1 < \cdots i_0^{m+n}.$$

Then the corollary holds with U generated by $\mathfrak{r}^1, \ldots, \mathfrak{r}^m$ and V generated by $\mathfrak{r}^{m+1}, \ldots, \mathfrak{r}^{m+n}$.

This enables us to prove [1]

THEOREM 11. *Let* $m, n \in N, p \in P, \mathbf{A} \in Z_{m,n}$. $\mathbf{A}$ *is partition regular in* Z_p^ω iff $\mathbf{A}$ *has the* p*-column property.*

Proof 11. Assume that $\mathbf{A}$ has the p-column property. Let $\mathbf{A}_i$ $(i = 1, \ldots, m)$ be the submatrices of $\mathbf{A}$, the columns of $\mathbf{A}_i$ being in classes with indices in

$\{1, \ldots, i\}$. By induction on i we show that for all i the matrix $\mathbf{A}_i$ is partition regular in Z_p^ω. This certainly holds for $\mathbf{A}_1$ as

$$\mathbf{A}_1 \cdot \begin{pmatrix} \mathfrak{r} \\ \vdots \\ \mathfrak{r} \end{pmatrix} = \mathbf{0} \qquad \text{for every} \quad \mathfrak{r} \in Z_p^\omega .$$

So let us assume that $\mathbf{A}_{i-1}$ $(i \geqslant 2)$ is partition regular in Z_p^ω. By induction on k we show that $\mathbf{A}_i$ is k-partition regular in Z_p^ω for each k. This certainly holds for $k = 1$. So assume that $\mathbf{A}_i$ is $(k-1)$-partition regular in Z_p^ω where n is determined by Lemma 1. So $\mathbf{A}_i$ is $(k-1)$-partition regular in every n-dimensional subspace V of Z_p^ω.

Let f be a k-partition of Z_p^ω. By Lemma 1 and hypothesis on $\mathbf{A}_{i-1}$ we know that $\mathbf{A}_{i-1}$ is k-partition regular in every m-dimensional subspace U of Z_p^ω for some m. By Corollary 10 we may assume that $f(\mathfrak{r} + \mathfrak{y}) = f(\mathfrak{r})$ whenever $\mathfrak{r} \in U$, $\mathfrak{y} \in V$, and $U \cap V = \mathbf{0}$.

Let $\mathfrak{r}^1, \ldots, \mathfrak{r}^r$ be a solution of $\sum_{\mathbf{A}_{i-1}} a^i \mathfrak{r}^i = \mathbf{0}$ satisfying $f(\mathfrak{r}^1) = f(\mathfrak{r}^2) = \cdots = f(\mathfrak{r}^r)$. By definition of the p-column property we have

$$\sum_{\mathbf{A}_{i-1}} \mathfrak{a}^i b_i + \sum_{\mathbf{A}_i \setminus \mathbf{A}_{i-1}} \mathfrak{a}^i b = \mathbf{0} \pmod{p}.$$

Hence

$$\sum_{\mathbf{A}_{i-1}} \mathfrak{a}^i (\mathfrak{r}^i + b_i \mathfrak{y}) + \sum_{\mathbf{A}_i \setminus \mathbf{A}_{i-1}} \mathfrak{a}^i b \mathfrak{y} = \mathbf{0} \quad \text{in } Z_p^\omega$$

for all $\mathfrak{y} \in V$. By assumption on U, V we have $f(\mathfrak{r}^i + b_i \mathfrak{y}) = f(\mathfrak{r}_i) = f(\mathfrak{r}_1)$ for all $i \in \{1, \ldots, r\}$.

We distinguish two cases:

Case 1. There exists $\mathfrak{y}_0 \in V$ with $f(\mathfrak{y}_0) = f(\mathfrak{r}_1)$. Then

$$\{\mathfrak{r}^1 + b_1 \mathfrak{y}_0 , \mathfrak{r}^2 + b_2 \mathfrak{y}_0 , \ldots, \mathfrak{r}^r + b_r \mathfrak{y}_0 , \mathfrak{y}_0\}$$

is a solution of $\mathbf{A}_i \mathbf{X} = \mathbf{0}$ contained in one class of f.

Case 2. For all $\mathfrak{y} \in V : f(\mathfrak{y}) \neq f(\mathfrak{r}_1)$. Then the restriction of f to V is a $(k-1)$-partition, and as $\mathbf{A}_i$ is $(k-1)$-partition regular in V, there is a solution contained in one class of f.

Assume now that $\mathbf{A}$ is partition regular in Z_p^ω. Define a p-partition f of Z_p^ω by $f(\mathfrak{r}) = b$ where $\mathfrak{r} = b\mathfrak{r}^*$, $\mathfrak{r}^*$ being a standard vector in Z_p^ω. By evaluating each coordinate it is easily established that $\mathbf{A}$ has the p-column property.

This same argument may be used to obtain a generalization of a theorem by Feise [3].

THEOREM 12. *Let* $\alpha_1, \alpha_2, r_1, r_2, m, n \in N, p \in P$, *and* $\mathbf{A} \in Z_{m,n}$. *If* $\mathbf{A}$ *is p-partition regular in* $G = Z^{r_1}_{p^{\alpha_1}} \oplus Z^{r_2}_{p^{\alpha_2}}$ then $\mathbf{A}$ *has the p-column property.*

Remark. The restriction to two summands is obviously unimportant, but made here in order to simplify notation.

Proof 12. Let $f(\mathfrak{r}) = R_p(b)$, where $\mathfrak{r} = b\mathfrak{r}^*$, $\mathfrak{r}^*$ being a standard element. Let $\mathfrak{r}_1 = (x_1^1, \ldots, x_1^{r_1+r_2}), \ldots, \mathfrak{r}_m = (x_m^1, \ldots, x_m^{r_1+r_2})$ be a solution of $\mathbf{AX} = \mathbf{0}$ contained in one class of f. Let $\{n_{\mathfrak{r}_i} \mid i = 1, \ldots, m\} = \{n_1, \ldots, n_t\}$ $(n_1 < n_2 < \cdots < n_t)$ be the set of the leading indices of the solution. Let $\mathbf{A}_j$ $(j = 1, \ldots, t)$ be the submatrix of $\mathbf{A}$ generated by the columns $\mathfrak{a}^i$ with $n_{\mathfrak{r}_i} \in \{n_1, \ldots, n_j\}$. By induction on j it is established that all $\mathbf{A}_j$ have the p-column property.

In order to see this for $\mathbf{A}_1$ calculate the n_1th coordinate and obtain

$$\sum_{n_{\mathfrak{r}_i} = n_1} \mathfrak{a}^i x_i^{n_1} = \mathbf{0} \pmod{p^\alpha}, \quad \alpha = \alpha_1 \text{ or } \alpha = \alpha_2. \tag{3}$$

For $x_i^{n_1}$ with $n_{\mathfrak{r}_i} = n_1$, we have by definition of f: $x_i^{n_1} = p^{\beta_i}(pk_i + l)$ for certain $\beta_i < \alpha, k_i, l \in \{1, \ldots, p-1\}$. Let $\{\beta_i \mid n_{\mathfrak{r}_i} = n_1\} = \{\beta_1, \ldots, \beta_s\}$ and $\beta_1 < \beta_2 < \cdots \beta_s$. So (3) becomes

$$p^{\beta} \sum_{\substack{n_{\mathfrak{r}_i} = n_1 \\ \beta_i = \beta_1}} \mathfrak{a}^i (pk_i + l) + \sum_{\substack{n_{\mathfrak{r}_i} = n_1 \\ \beta_i > \beta_1}} \mathfrak{a}^i p^{\beta_i} (pk_i + l) = \mathbf{0} \pmod{p^\alpha}. \tag{4}$$

So

$$\sum_{\substack{n_{\mathfrak{r}_i} = n_1 \\ \beta_i = \beta_1}} \mathfrak{a}^i l = \mathbf{0} \pmod{p}.$$

For $j > 1$ the p-column property is established by similar calculations.

5. Partition Regular Systems

Here we give a complete description of the matrices which are partition regular in a given group G.

DEFINITION. Let G be a group. Let $\sum(G)$ be the set of all $p \in P^*$ such that G contains infinitely many elements with order p. If G contains elements of arbitrary large finite order then include ∞ in $\sum(G)$.

THEOREM 13. *Let* $m, n \in N$, $\mathbf{A} \in Z_{m,n}$, *and* G *be a group.* $\mathbf{A}$ *is partition regular in* G iff $\Gamma(\mathbf{A}) \cap \sum(G) \neq \varnothing$ *or there exists* $x \in G \backslash \{0\}$ *such that* $\mathbf{A}(x, \ldots, x)^T = \mathbf{0}$.

Proof 13. Let $p \in \sum(G) \cap \Gamma(\mathbf{A})$ and $p \in P$. Then $\mathbf{A}$ has the p-column property and $Z_p^\omega \subset G$. So $\mathbf{A}$ is partition regular in G.

Let $\infty \in \sum(G) \cap \Gamma(\mathbf{A})$. Then $\mathbf{A}$ has column property and G contains cyclic subgroups of arbitrary large order. So $\mathbf{A}$ is partition regular in G.

Now let $\mathbf{A}$ be partition regular in G. Let k_0 be large enough. Then $\mathbf{A}$ is k_0-partition regular in a finitely generated subgroup $G_0 = \bigoplus Z^{n_i}_{p_i^{\alpha_i}}$ $(p_i \in P^*)$. By Lemma 2 we may assume that $\mathbf{A}$ is k_1-partition regular (k_1 large enough) in the subgroup G_1 of G_0 with $p_i < p_{\mathbf{A}}$ for all i, or k_1-partition regular in the subgroup G_2 of G_0 with $p_i \geqslant p_{\mathbf{A}}$ for all i.

Case 1. $\mathbf{A}$ is k_1-partition regular in G_1. By Lemma 2 $\mathbf{A}$ is k_2-partition regular in the subgroup $G_1^1 = \bigoplus Z^{n_i}_{p_i^{\alpha_i}}$ of G_1 with $p_i \notin \sum(G)$ or k_2-partition regular in the subgroup of $G_1^2 = \bigoplus Z^{n_i}_{p_i^{\alpha_i}}$ with $p_i \in \sum(G)$. In the first case there are only finitely many elements x in G such that ord x is a power of p_i. So for k_2 large enough we can assume that each element of G_1^1 is in a class by itself. Hence there exists $x \in G_1^1 \subset G$ such that $\mathbf{A}(x,..., x)^T = \mathbf{0}$. In the second case $\mathbf{A}$ is p-partition regular in $\bigoplus Z^{n_i}_{p^{\alpha_i}}$ for some $p \in \sum(G)$. Hence $\mathbf{A}$ has the p-column property and $Z_p^\omega \subset G$ whence

$$\Gamma(\mathbf{A}) \cap \sum(G) \neq \varnothing .$$

Case 2. $\mathbf{A}$ is k_1-partition regular in G_2. Hence $\mathbf{A}$ has the column property. We may assume that $\sum(G) \subset \{2,..., p_{\mathbf{A}} - 1\}$ for otherwise we are done. Hence the set of orders of elements of G is bounded. Hence the cardinality of G_2 is bounded by a bound depending on $G\mathbf{A}$ only. So for k_1 large enough every element of G_2 can be put into a class by itself and there is $x \in G_2 \subset G$ such that $\mathbf{A}(x,..., x)^T = \mathbf{0}$.

Obviously k_0 may be defined such that the above considerations hold.

6. Universal Sets

If A, B are matrices which are partition regular in Z then by Theorem 5

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$$

has the same property as $\infty \in \Gamma(\mathbf{C}) = \Gamma(\mathbf{A}) \cap \Gamma(\mathbf{B})$. This is the starting point for an interesting partition theory of Z. However by Theorem 11 the above remark does not hold for arbitrary groups as

$$\Gamma(\mathbf{A}) \cap \Gamma(\mathbf{B}) \cap \sum(G) = \varnothing$$

is possible while $\Gamma(\mathbf{A}) \cap \sum(G) \neq \varnothing$ and $\Gamma(\mathbf{B}) \cap \sum(G) \neq \varnothing$.

DEFINITION. Let $p \in P^*$ and $M(p)$ be the set of all finite matrices with entries in Z having the p-column property.

Remark. If $\mathbf{A}, \mathbf{B} \in M(p)$ are partition regular in a group G, then

$$C = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$$

is partition regular in G.

DEFINITION. Let $p \in P^*$ and G be a group. A subset X of $G \backslash \{0\}$ is p-universal if for every $\mathbf{A} \in M(p)$ the system

$$\mathbf{A} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \mathbf{0}$$

has a solution in X.

The following lemma shows the interest in universal sets.

LEMMA 14. *Let $p \in P^*$ and G be a group such that $p \in \sum(G)$. Let f be a partition of $G \backslash \{0\}$ into finitely many classes. Then one of the classes is p-universal.*

Proof 14. Let f be a k-partition of $G \backslash \{0\}$, and assume that none of the classes is p-universal. Hence for $i = 1, \ldots, k$ there exist matrices $\mathbf{A}_i \in M(p)$ such that $\mathbf{A}_i \mathbf{X} = \mathbf{0}$ has no solution in the class with index i.

Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ & \mathbf{A}_2 & & \vdots \\ \mathbf{0} & & \ddots & \vdots \\ \vdots & & & \\ \mathbf{0} & \cdots & \cdots & \mathbf{A}_k \end{pmatrix}.$$

Then $\Gamma(\mathbf{A}) \cap \sum(G) \neq \varnothing$. So there exists $j \in \{1, \ldots, k\}$ such that $\mathbf{AX} = \mathbf{0}$ has a solution in the class with index j. But then $\mathbf{A}_j \mathbf{X} = \mathbf{0}$ has a solution in the class with index j, a contradiction.

From here on we have to distinguish two cases: $p = \infty$ and $p \in P$. This distinction is necessary as Z_p is a field, while Z is not. The theory of ∞-universal sets has been developed in [2], so we omit this here.

Next we give another characterization of p-universal sets.

LEMMA 15. *For every $\mathbf{A} \in M(p)$ there exists $m \in N$ such that every m-set in Z_p^ω contains a solution of $\mathbf{AX} = \mathbf{0}$.*

Proof 15. Let $\mathbf{A} \in Z_{m,n} \cap M(p)$. Hence there exists $f: \{1,..., m\} \to \{1,..., m\}$ such that for all j

$$\sum_{f(i)=j} \mathfrak{a}^i + \sum_{f(i)<j} \mathfrak{a}^i b_i^j = \mathbf{0} \pmod p$$

for some $b_i^j \in Z_p$. Let $\mathbf{A}_j$ be the submatrix of $\mathbf{A}$ generated by $\{\mathfrak{a}^i \mid f(i) \leqslant j\}$. Clearly $\mathbf{A}_j \in M(p)$ for every j. By induction on j it is shown that every j-set contains a solution of $\mathbf{A}_j\mathbf{X} = \mathbf{0}$. This certainly holds for $\mathbf{A}_1$ as every $x \in Z_p^\omega$ is a solution. Let $(\mathfrak{r}_1, ..., \mathfrak{r}_j)$ be a j-basis. Then we may assume that $(\mathfrak{r}_1, ..., \mathfrak{r}_{j-1})_p$ contains a solution of $\mathbf{A}_{j-1}\mathbf{X} = \mathbf{0}$, i.e.

$$\sum_{f(i)<j} \mathfrak{a}^i \mathfrak{r}_i{}^* = \mathbf{0} \quad \text{in } Z_p^\omega.$$

As $\mathbf{A}_j \in M(p)$, we have

$$\sum_{f(i)=j} \mathfrak{a}^i + \sum_{f(i)<j} \mathfrak{a}^j b_i^j = \mathbf{0} \pmod p.$$

Hence

$$\sum_{f(i)=j} \mathfrak{a}^i \mathfrak{r}_j + \sum_{f(i)<j} \mathfrak{a}^i(\mathfrak{r}_i{}^* + b_i^j \mathfrak{r}_j) = \mathbf{0} \quad \text{in } Z_p^\omega.$$

But $\{\mathfrak{r}_j\} \cup \{\mathfrak{r}_i{}^* + b_i^j\mathfrak{r}_j \mid f(i) < j\} \subset (\mathfrak{r}_1, ..., \mathfrak{r}_j)_p$.

LEMMA 16. *For every $m \in N$ there exists $\mathbf{A} \in M(p)$ such that every solution $\{\mathfrak{r}_1, ..., \mathfrak{r}_s\} \subset Z_p^\omega \backslash \{0\}$ of $\mathbf{AX} = \mathbf{0}$ contains an m-set.*

Proof 16. Induction on m. For $m = 1$ let $\mathbf{A} = (1, -1)$. Assume now that every solution of $\mathbf{A}_{m-1}\mathbf{X} = \mathbf{0}$ contains an $(m-1)$-set, $(\mathbf{A}_{m-1} \in Z_{u,v})$. Consider the following system of equations.

$$\mathbf{A}_{m-1} \begin{pmatrix} x_1 \\ \vdots \\ x_u \end{pmatrix} = \mathbf{0}, \tag{5}$$

$$x_j + \lambda x_{u+1} - x_{\lambda,j} = 0 \qquad (j = 1,..., u, \quad \lambda = 1,..., p-1).$$

Let $\mathbf{A}_m$ be the matrix of (5). Obviously $\mathbf{A}_m \in M(p)$ if $\mathbf{A}_{m-1} \in M(p)$. Let $X = \{\mathfrak{r}_1, ..., \mathfrak{r}_{u+1}\} \cup \{\mathfrak{r}_{\lambda,j} \mid 1 \leqslant j \leqslant s, 1 \leqslant \lambda \leqslant p-1\}$ be a solution of $\mathbf{AX} = \mathbf{0}$ in $Z_p^\omega \backslash \{0\}$.

By rearranging the columns of $\mathbf{A}_{m-1}$ we may assume that $(\mathfrak{r}_1, ..., \mathfrak{r}_{m-1})$ is an $m-1$-basis for an $m-1$-set contained in $\{\mathfrak{r}_1, ..., \mathfrak{r}_u\}$. We show that $(\mathfrak{r}_1, ..., \mathfrak{r}_{m-1}, \mathfrak{r}_{u+1})_p$ is an m-set contained in X. Indeed $(\mathfrak{r}_1, ..., \mathfrak{r}_{u+1})_p \subset X$. In order to show the linear independence of $(\mathfrak{r}_1, ..., \mathfrak{r}_{u+1})$ assume that

$$\mathfrak{r}_{u+1} = \sum_{i=1}^{m-1} \mu^i \mathfrak{r}_i \qquad \text{and not all} \quad \mu^i = 0.$$

Let $t \in \{1,..., p-1\}$ be such that $t\mathfrak{r}_{u+1} \in (\mathfrak{r}_1,..., \mathfrak{r}_{m-1})_p$. Hence $t\mathfrak{r}_{u+1} = \mathfrak{r}_j$ for some $j \in \{1,..., u\}$, and $\mathbf{0} = \mathfrak{r}_j + (p-t)\,\mathfrak{r}_{s+1} = \mathfrak{r}_{p-t,j}$. So the solution of $\mathbf{A}_m\mathbf{X} = \mathbf{0}$ was not contained in $Z_p^\omega \backslash \{0\}$, a contradiction.

THEOREM 17. *Let G be a group and $p \in P$. A subset X of G is p-universal* iff *for every $m \in N$ there exists an m-set contained in X.*

Proof 17. If X is p-universal then by Lemma 16 X contains an m-set for each m. If X contains an m-set for each m then by Lemma 15 X is p-universal.

In [9] Rado asked whether the following is true. Let X be an ∞-universal set in Z and f a partition of X into finitely many classes. Then one of the classes is ∞-universal again. In short, is ∞-universality hereditary? This was shown to be true in [2] by using (m, p, c)-sets. Here we shall show that this holds for p-universality as well. In fact there is a full Ramsey theorem for m-sets. Note that a Ramsey theorem for (m, p, c)-sets has been proved by Leeb [7, 8].

THEOREM 18. *Let $m, n, k \in N, p \in P$. Then there is $r \in N$ such that for every k-partition of the n-subsets of an r-set X in Z_p^ω there is an m-subset of X with all its n-subsets in the same class.*

Proof 18. We need the following Ramsey theorem for finite vector spaces [5] by Graham, Leeb, and Rothschild.

THEOREM 19. *Let $m, n, k \in N, p \in P$. Then there exists $r \in N$ such that for every k-partition of the n-dimensional subspaces of Z_p^r there is an m-dimensional subspace with all its n-dimensional subspaces in the same class.*

Let $m, n, k \in N, p \in P$ and r be defined by Theorem 19. Let X be an r-set and f be a k-partition of the n-subsets of X. We define a k-partition f' of the n-dimensional subspaces of Z_p^ω. $f'L(x_1,..., x_n)_p = f(x_1,..., x_n)_p$. Now Theorem 18 follows by Theorem 19, the transitivity of the relation "n-subset of" and the fact that every space has a basis consisting of standard vectors.

COROLLARY 20. *Let X be a p-universal set ($p \in P$) and f a partition of X into finitely many classes. Then one of the classes is p-universal.*

Proof 20. Assume that none of the classes of the k-partition f is p-universal. By Theorem 17 for $i = 1,..., k$ there exists m_i such that the class with index i does not contain an m_i set. Let $m = \max_{1 \leqslant i \leqslant k} m_i$. Apply Theorem 18 with $m, n = 1, k, p$, and obtain an m-set contained in one class of f, say class i. As $m_i \leqslant m$ class i contains an m_i-set, a contradiction.

7. Problems

—Develop a theory of infinite partition regular matrices [6].

—Rado [9]. In Lemma 4, $p_{\mathbf{A}}$ depends on **A** and not only on the number m of columns in **A**. Is that the best possible?

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